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A hyperbolic equation for turbulent diffusion

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Abstract. A hyperbolic equation, analogous to the telegrapher's equation in one dimension, is introduced to describe turbulent diffusion of a passive additive in a turbulent flow. The predictions of this equation, and those of the usual advection–diffusion equation, are compared with data on smoke plumes in the atmosphere and on heat flow in a wind tunnel. The predictions of the hyperbolic equation fit the data at all distances from the source, whereas those of the advection–diffusion equation fit only at large distances. The hyperbolic equation is derived from an integrodifferential equation for the mean concentration which allows it to vary rapidly. If the mean concentration varies sufficiently slowly compared with the correlation time of the turbulence, the hyperbolic equation reduces to the advection–diffusion equation. However, if the mean concentration varies very rapidly, the hyperbolic equation should be replaced by the integrodifferential equation.

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1. Introduction

Turbulent diffusion of a passive additive in an incompressible fluid is usually described by the parabolic advection–diffusion equation, proposed by Boussinesq (1877) and Taylor (1915),

$$\frac{\mathrm{d}c}{\mathrm{d}t} - \nabla \cdot [(K + D)\nabla c] = g. \tag{1.1}$$

Here $c\,(\boldsymbol{x}, t)$ is the concentration, K is the molecular diffusion coefficient, $D(\boldsymbol{x}, t)$ is the turbulent diffusion coefficient, g is the strength of the source of additive, $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \boldsymbol{u} \cdot \nabla$ is the advective time derivative and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the mean velocity of the fluid.

Batchelor and Townsend (1956), p 360, suggest that 'a description of the diffusion by some kind of integral equation is more to be expected'. We derive such an equation, and from it we derive the following hyperbolic equation for c:

$$T\frac{\mathrm{d}^2 c}{\mathrm{d}t^2} + \frac{\mathrm{d}c}{\mathrm{d}t} - \nabla \cdot [(K + D)\,\nabla c] - T\nabla \cdot \left(\frac{\mathrm{d}D}{\mathrm{d}t}\nabla c\right) + D\nabla T \cdot \frac{\mathrm{d}}{\mathrm{d}t}\nabla c$$
$$+T\,\partial_k\,[D\nabla u_k \cdot \nabla c] = g + T\frac{\mathrm{d}g}{\mathrm{d}t}. \tag{1.2}$$

Here T is the correlation time of the turbulent velocity as defined by (4.15). When T is negligible, equation (1.2) reduces to (1.1). We obtain (1.2) by first deriving an

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integrodifferential equation for c, following Keller (1964, 1966, 1971) and Saffman (1969). From this equation, we obtain (1.2) when c does not vary appreciably during the correlation time T. When u, D and T are constant, or slowly varying, equation (1.2) simplifies to

$$T\frac{\mathrm{d}^2c}{\mathrm{d}t^2} + \frac{\mathrm{d}c}{\mathrm{d}t} - (K + D)\Delta c = g + T\frac{\mathrm{d}g}{\mathrm{d}t}. \tag{1.3}$$

This is a three-dimensional form of the telegrapher's equation. The one-dimensional form of (1.3), with $u = 0$ and $g = 0$, was derived by Goldstein (1950) for a correlated random walk along the x-axis.

We shall compare the predictions of (1.1) with experimental data and with the predictions of (1.2). The experimental data we use are those of Högstrom (1964) on smoke plumes in the atmosphere and those of Stapountzis *et al* (1986) on heat diffusion in a wind tunnel. We shall find solutions of (1.1) and (1.2) corresponding to these experiments, and evaluate them using appropriate values of D and T. We shall see that the solutions of (1.1) fit the data only at points far from the source, whereas the solutions of (1.2) fit the data fairly well over the whole range of distances. We conclude that (1.2) provides a better description of turbulent diffusion than does (1.1) when c varies rapidly in space or time. If c varies extremely rapidly, the integrodifferential equation (4.11) in section 4 should be even more accurate than (1.2).

There has been a great deal of work on turbulent diffusion (see the treatise of Monin and Yaglom (1975)). One very useful procedure is that of describing the velocity of a diffusing particle by the Langevin equation. This equation has been used, in particular, by Durbin (1980), Sawford and Hunt (1986) and Thompson (1990). These authors have used it and extensions of it, to determine the variance and two-point correlation of c. Since the Langevin equation includes a Gaussian process for the forcing function, it leads to arbitrarily large particle velocities. Therefore, the diffusion equation (1.1), which follows from the Langevin equation, leads to diffusion with infinite speed. In the derivation of (1.2), infinite speed is avoided by using the first two moments of the velocity, and not specifying that the force or velocity is Gaussian. Anand and Pope (1985) have shown that for certain choices of the random functions in the Langevin equation, its solution provides a good fit to the plume width in the wind tunnel data of Stapountzis *et al* (1986). The fit is as good as that provided by (1.2) or even better. This result depends upon the assumptions made about the Lagrangian velocity functions in the Langevin equation. The derivation of (1.2) does not involve such assumptions, but just uses correlation functions of the Eulerian velocity.

Another experimental comparison of (1.1) and (1.3) could be made by determining experimentally the dispersion equation for time harmonic plane waves. For the wave $c = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$, equation (1.3) with $g = 0$ and constant coefficients yields

$$k^2 = (D + K)^{-1}(T\omega^2 + \mathrm{i}\omega). \tag{1.4}$$

Equation (1.1) yields (1.4) with $T = 0$. For ω of the order of T^{-1} the difference between the two predictions should be observable.

The method of derivation of the integro-differential equation for $\langle c\rangle$ in section 4 has been used to derive equations for $\langle u\rangle$ and for the two-point correlation function of u (Keller 1966). The resulting equations are similar to those obtained by the direct interaction approximation of Kraichnan (1959). This derivation provides a different way of considering closure approximations.

In section 2, solutions of (1.1) and (1.2) are obtained which correspond to the experimental situations. In section 3, these solutions are compared with the data from the two sets of experiments. The derivations of (1.2) and of more general equations are presented in section 4.

2. Diffusion from point and line sources

Let us consider the steady concentrations produced by a point source of strength g_0 at the origin and by a uniform line source of strength g_0 per unit length along the z-axis. In both cases we assume that the mean flow velocity is along the x-axis and is constant. Thus we write $\boldsymbol{u}(\boldsymbol{x}, t) = U\hat{\boldsymbol{x}}$ so that $\frac{\mathrm{d}}{\mathrm{d}t} = U\partial_x$. In the point source case $c = c(x, r)$ where $r = (y^2 + z^2)^{1/2}$, while in the line source case $c = c(x, y)$. We also let $D(\boldsymbol{x})$ and $T(x)$ depend upon x, and we neglect K compared with D.

When we use these assumptions in (1.2) for the line source, and divide by U, we obtain

$$\left[TU\partial_x^2 + \partial_x - U^{-1}\partial_x D\partial_x - \left(U^{-1}D + TD'\right)\partial_y^2 - T\partial_x D'\partial_x + DT'\partial_x^2\right]c(x, y) = U^{-1}g_0\delta(x)\delta(y) + g_0 T\delta'(x)\delta(y). \tag{2.1}$$

Here the prime denotes an x derivative. In the point source case we must replace $\partial_y^2 c(x, y)$ by $r^{-1}\partial_r r\partial_r c(x, r)$ and $\delta(y)$ by $\delta(r)/\pi r$ in (2.1).

In the line source case, the integrated concentration at x, $c_0(x)$, is defined by

$$c_0(x) = \int_{-\infty}^{\infty} c(x, y)\,\mathrm{d}y. \tag{2.2}$$

The second moment, $c_2(x)$, is defined by

$$c_2(x) = \int_{-\infty}^{\infty} c(x, y)y^2\,\mathrm{d}y. \tag{2.3}$$

Upon integrating (2.1) with respect to y, we obtain the following ordinary differential equation for $c_0(x)$:

$$\left[TU\partial_x^2 + \partial_x - U^{-1}\partial_x D\partial_x - T\partial_x D'\partial_x + DT'\partial_x^2\right]c_0(x) = U^{-1}g_0\delta(x) + g_0 T\delta'(x). \tag{2.4}$$

Then multiplying (2.1) by y^2, integrating over y and using integration by parts, we obtain an ordinary differential equation for $c_2(x)$:

$$\left[TU\partial_x^2 + \partial_x - U^{-1}\partial_x D\partial_x - T\partial_x D'\partial_x + DT'\partial_x^2\right]c_2(x) = 2(U^{-1}D + TD')c_0(x). \tag{2.5}$$

For the point source we define $c_0(x)$ and $c_2(x)$ by

$$c_0(x) = 2\pi\int_0^{\infty} c(x, r)r\,\mathrm{d}r \tag{2.6}$$

$$2c_2(x) = 2\pi\int_0^{\infty} r^2 c(x, r)r\,\mathrm{d}r. \tag{2.7}$$

With these definitions, we find from (2.1), with the replacements indicated just below it, that c_0 and c_2 again satisfy (2.4) and (2.5).

By setting $T = 0$ in (2.4) and (2.5), we obtain the equations for c_0 and c_2 corresponding to the advection–diffusion equation (1.1).

We shall solve (2.4) and (2.5) for $c_0(x)$ and $c_2(x)$, and compare the solutions with corresponding observed values. First we write (2.4) and (2.5) in the forms

$$\alpha c_0'' + \beta c_0' = \frac{g_0}{U}\delta(x) + g_0 T\delta'(x) \tag{2.8}$$

$$a c_2'' + \beta c_2' = \gamma c_0(x). \tag{2.9}$$

Here $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are defined by

$$\begin{aligned} \alpha &= UT - D/U - TD' + DT' \\ \beta &= 1 - D'/U - TD'' \\ \gamma &= 2(D/U + TD'). \end{aligned} \tag{2.10}$$

We also introduce the integrating factor

$$P(x) = \exp\left[-\int_0^x \frac{\beta(s)}{\alpha(s)}\,\mathrm{d}s\right]. \tag{2.11}$$

We seek solutions in which $c_0(x)$ and $c_2(x)$ tend to zero as $x \to -\infty$, and which grow less rapidly than exponential as $x \to +\infty$. The nature of the solutions depends upon the sign of $\alpha(x)/\beta(x)$ for large positive x and for large negative x. When $\alpha/\beta < 0$ for all $|x|$ larger than some x_0,

$$c_0(x) = g_0 \begin{cases} \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T}{P\alpha}\right) - \frac{1}{P\alpha U}\right]_{x=0} \int_{-\infty}^x P(\xi)\,\mathrm{d}\xi & x < 0 \\ \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T}{P\alpha}\right) - \frac{1}{P\alpha U}\right]_{x=0} \int_{-\infty}^0 P(\xi)\,\mathrm{d}\xi + \frac{T(0)}{\alpha(0)} & x \geqslant 0 \end{cases} \tag{2.12}$$

$$c_2(x) = -\int_{-\infty}^x P(\xi) \int_\xi^\infty \frac{\gamma(s)c_0(s)}{\alpha(s)P(s)}\,\mathrm{d}s\,\mathrm{d}\xi. \tag{2.13}$$

When $\alpha/\beta > 0$ for all $|x|$ larger than some x_0,

$$c_0(x) = g_0 \begin{cases} 0 & x < 0 \\ -\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T}{P\alpha}\right) - \frac{1}{P\alpha U}\right]_{x=0} \int_0^x P(\xi)\,\mathrm{d}\xi + \frac{T(0)}{\alpha(0)} & x \geqslant 0 \end{cases} \tag{2.14}$$

$$c_2(x) = \begin{cases} 0 & x < 0 \\ \int_0^x P(\xi) \int_0^\xi \frac{\gamma(s)c_0(s)}{\alpha(s)P(s)}\,\mathrm{d}s\,\mathrm{d}\xi & x \geqslant 0. \end{cases} \tag{2.15}$$

For the atmospheric diffusion experiments to be considered in section 3, we assume that the turbulence is uniform, so we take D and T to be constants. Then (2.10) shows that $\alpha = UT - D/U$, $\beta = 1$ and $\gamma = 2D/U$. With these values, the solutions (2.12)–(2.15) simplify to the following.

When $\alpha < 0$, i.e. $T < D/U^2$,

$$c_0(x) = \begin{cases} \frac{-g_0 D}{U^2\alpha}\,\mathrm{e}^{-x/\alpha} & x < 0 \\ \frac{g_0}{U} & x \geqslant 0 \end{cases} \tag{2.16}$$

$$c_2(x) = \frac{2g_0 D}{U^2} \begin{cases} \left[\frac{D}{U} - \alpha + \frac{D}{U\alpha}x\right]\mathrm{e}^{-x/a} & x < 0 \\ \left[x + \frac{D}{U} - \alpha\right] & x \geqslant 0. \end{cases} \tag{2.17}$$

When $\alpha > 0$, i.e. $T > D/U^2$,

$$c_0(x) = \frac{g_0}{U}\begin{cases} 0 & x < 0 \\ \left[1 + \dfrac{D}{\alpha U}\mathrm{e}^{-x/\alpha}\right] & x \geqslant 0. \end{cases} \tag{2.18}$$

$$c_2(x) = \frac{2g_0 D}{U^2}\begin{cases} 0 & x < 0 \\ \left[x + \left(\dfrac{D}{U} - \alpha\right)\left(1 - \mathrm{e}^{-x/\alpha}\right) - \dfrac{D}{\alpha U}x\mathrm{e}^{-x/\alpha}\right] & x \geqslant 0. \end{cases} \tag{2.19}$$

For the advection–diffusion equation (1.1), $T = 0$ so (2.16) and (2.17) apply with $\alpha = -D/U$.

For the wind tunnel experiments to be considered in section 4, the turbulence is not constant so we shall need the solutions (2.12)–(2.15).

3. Comparison with experiments

3.1. Atmospheric measurements

Many measurements of the spread of smoke plumes downwind of a point source have been made. We will use those of Högstrom (1964), who introduced innovations which permitted him to obtain photographic traces of the plume for a much longer time than had been possible before. As a result, he recorded both the initial linear growth of the plume width, and the slower parabolic growth far downstream. The experiments were performed near the towns of Studsvik and Ågesta, Sweden. Since Studsvik is on the coast, the data from there are complicated by the effect of the topographical discontinuity on the turbulence. Therefore, we use only the Ågesta data.

The data for the vertical width $\sigma(x)$ of the plume, as a function of distance x downstream from the source, are taken from table 3, p 220 of Högstrom's paper. Results of the three separate experiments listed in table 3 are shown as symbols in figure 1 after they have been made comparable by multiplication by a factor to remove the effect of stratification in the manner described by Högstrom. The mean wind profile was measured by means of six anemometers mounted on a mast. Fitting the data to well known empirical formulae for atmospheric boundary layers yielded the following values for the mean and rms turbulent velocities: $U = 5.92$ m s^{-1} and $\sqrt{\langle \boldsymbol{u}'^2\rangle}/U = 0.12$.

The theoretical value of $\sigma(x)$ is defined by the equation

$$\sigma^2(x) = \frac{c_2(x)}{c_0(x)}. \tag{3.1}$$

The values of c_0 and c_2 are given by (2.16) and (2.17) for the advection–diffusion equation (1.1), and by (2.18) and (2.19) for the hyperbolic equation (1.2).

To compare the data with the theory, we first solve (4.7) for $\boldsymbol{y}$, with $\langle \boldsymbol{u}\rangle = \boldsymbol{U}$, to obtain $\boldsymbol{y}(s, t, \boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{U}(s - t)$. We use this in (4.13) for D_{ij} and assume that the correlation function has the form $C_{ij}\left[\boldsymbol{x}, t, \boldsymbol{x} + \boldsymbol{U}(s - t), s\right] = \langle \boldsymbol{u}'^2\rangle \delta_{ij} f\left(\frac{s-t}{T}\right)$. Then (4.13) shows that $D_{ij} = D\delta_{ij}$ where D is given by

$$D = \langle \boldsymbol{u}'^2\rangle T \int_0^\infty f\left(\frac{s-t}{T}\right)\frac{\mathrm{d}s}{T}. \tag{3.2}$$

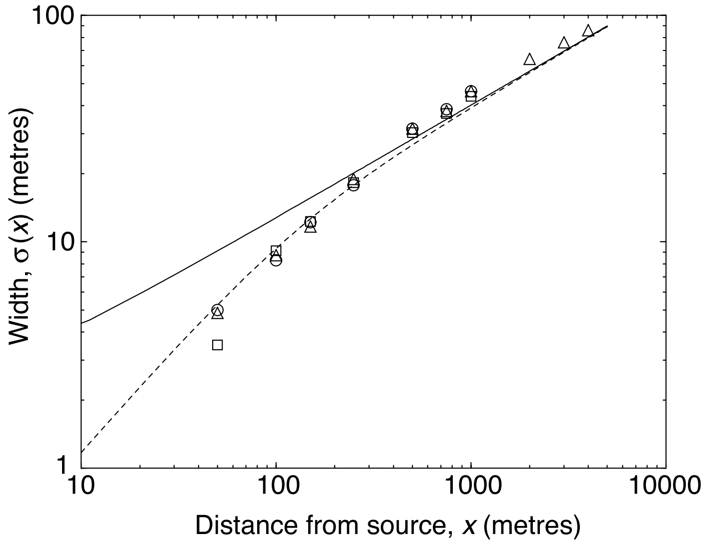

Figure 1. Width $\sigma(x)$ of a smoke plume in the atmosphere as a function of distance x downstream from the source, both measured in metres. The symbols represent the three experiments listed in table 3 on p 220 of Högstrom (1964) after correcting for the effect of stratification. The full curve is computed from (3.3), which is obtained from the solution of the advection–diffusion equation (1.1). The broken curve is computed from (3.1) using the solutions (2.18) and (2.19) of the hyperbolic equation (1.2).

We have set the upper limit $t/T = \infty$ because the turbulence was initiated far upstream, so it is fully developed. The integral in (3.2) is a numerical constant which we assume to have the value one, so that $D = \langle \boldsymbol{u}'^2 \rangle T$. This will be the case, for example, if $f(\xi) = \mathrm{e}^{-\xi}$. Then (4.15) shows that $T_{ij} = \delta_{ij} T$ provided that $\int_0^\infty f(\xi)\, \xi \, \mathrm{d}\xi = 1$, which is also the case for $f(\xi) = \mathrm{e}^{-\xi}$.

To determine D we use (3.1) with c_0 and c_2 given by (2.16) and (2.17) with $\alpha = -D/U$, to obtain

$$\sigma^2(x) = \frac{c_2(x)}{c_0(x)} = \frac{2D}{U}\left(x + \frac{2D}{U}\right). \tag{3.3}$$

A plot of the measured values of $\sigma^2(x)$ versus x shows that the last few data points for large x lie in the asymptotic regime where the slope $\mathrm{d}\sigma^2/\mathrm{d}x$ is constant. Measurement from the graph gives $\mathrm{d}\sigma^2/\mathrm{d}x = 1.612$ m. Equating this to the theoretical slope $2D/U$ given by (3.3) yields $D = 1.612U/2 = 4.77$ m^2 s^{-1}. We use this value in (3.3) to compute $\sigma^2(x)$ for the advection–diffusion equation (1.1), and the result is shown by the full curve in figure 1.

To evaluate $\sigma^2(x)$ for the hyperbolic equation (1.2), we need the value of T. To get it we use the relation $D = \langle \boldsymbol{u}'^2 \rangle T$ given by (3.2), with the measured values $\sqrt{\langle \boldsymbol{u}'^2 \rangle}/U = 0.12$, $U = 5.95$ m s^{-1}, and the value $D = 4.77$ m^2 s^{-1} determined above. This yields $T = 9.45$ s. Since $D/U^2 = 0.14$ s, we see that $T \gg D/U^2$. The derivative terms in (2.10) are small, so $\alpha \approx UT - D/U = U(T - D/U^2)$ and $\beta \approx 1$, and therefore $\alpha/\beta > 0$. Consequently, equations (2.18) and (2.19) apply, and we use them in (3.1) to get $\sigma^2(x)$. The result is shown by the broken curve in figure 1.

The figure shows that both the advection–diffusion equation and the hyperbolic equation give results consistent with the data at large distances from the source. However, closer to the source, the hyperbolic equation continues to give fairly accurate results, whereas the prediction of the advection–diffusion equation differs significantly from the experimental data. This is due to the different behaviour predicted by the advection–diffusion and the hyperbolic equations as $x \to 0$. The telegraph equation with $\alpha > 0$ gives the correct behaviour $\sigma(x)$ proportional to x for x small, whereas the advection–diffusion equation gives $\sigma(x) \sim 2D/U$. Far downstream, where $x \to \infty$, both equations predict $\sigma(x)$ to be proportional to $\sqrt{x}$ in agreement with the data. The failure of the diffusion approximation close to the source is to be expected. It is significant that finite correlation time effects, embodied in the hyperbolic equation, lead to much improved agreement with the data.

3.2. Wind tunnel experiments

In the experiments of Stapountzis *et al* (1986), turbulence is generated by passing air through a grid. The 'passive scalar' is the temperature variation produced by a fine electrically heated wire stretched across the test section of the tunnel. The wire is so thin that its wake does not alter the turbulence significantly. Molecular diffusion is small; its effect can be accounted for approximately by first measuring the wake in the absence of turbulence, and using this measurement to correct the data with turbulence. The mean velocity and turbulent intensities are substantially constant across the tunnel except in the boundary layers. The experiment is set up so that the turbulent thermal wake never reaches these layers. Therefore, the theory of turbulent diffusion from a line source in a uniform mean flow can be expected to hold. The rigorously controlled laboratory environment permits the theoretical idealizations to be realized more accurately than is possible in the atmospheric diffusion experiments.

The following properties of the wind tunnel turbulence were measured:

$$\langle \boldsymbol{u}'^2(x)\rangle = 0.15\left(\frac{x}{M} + \frac{x_0}{M}\right)^{-1.43} U^2 \tag{3.4}$$

$$\frac{\lambda(x)}{M} = 0.0703\left(\frac{x}{M} + \frac{x_0}{M}\right)^{0.47}. \tag{3.5}$$

Here $U = 4.35$ m s^{-1} is the mean speed of the air in the wind tunnel, $M = 0.0254$ m (1 inch) is the wire spacing in the grid generating the turbulence, $x_0 = 19.3M$ is the distance from the source, at $x = 0$, to the grid, and $\lambda(x)$ is the integral scale, measured using the Taylor hypothesis of 'frozen turbulence'. The turbulence exhibits a slight anisotropy so that the turbulent intensities and scales in the streamwise and cross-stream directions are not precisely equal. Here we have taken the values corresponding to the cross stream direction for the following reason: in the anisotropic case D is a tensor with principal directions along the coordinate axes and in this case, if the derivations of (2.4) and (2.5) are carried through, the component D_{yy} would take the place of D.

To determine $D(x)$ and $T(x)$ we proceed as before to obtain $D(x) = \langle u'^2(x)\rangle T(x)$. We expect $T(x)$ to be expressible in terms of $\lambda(x)$ and $\langle u'^2(x)\rangle$, but we have not found any measurements relating them. Therefore, we make the plausible assumption that $T(x)$ is given by

$$T(x) = C\lambda(x)/\sqrt{\langle \boldsymbol{u}'^2(x)\rangle}. \tag{3.6}$$

Here C is a dimensionless parameter which is chosen to make the theory fit the data at the point furthest downstream. The proper value of C was 0.7 for the advection–diffusion equation and 1.95 for the hyperbolic equation. By using (3.5) in (3.6), and in $D = \langle u'^2\rangle T$, we obtain the

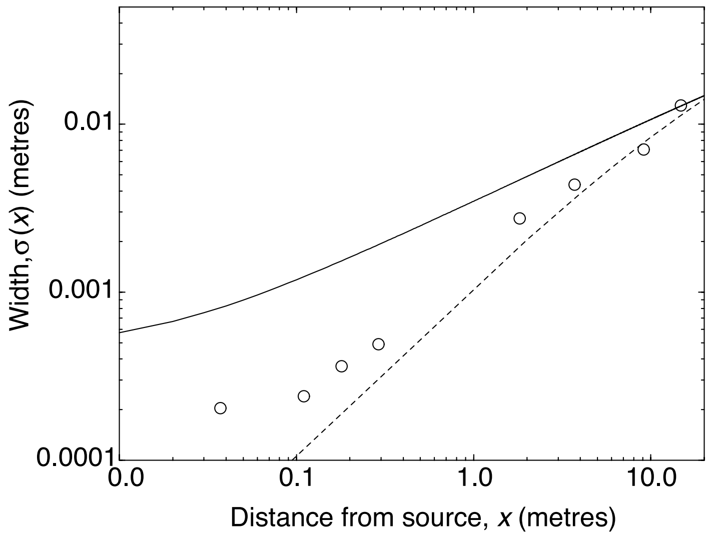

Figure 2. Width $\sigma(x)$ of a thermal plume in a wind tunnel as a function of the distance x from the source, both measured in metres. The open circles are based upon the data of Stapountzis *et al* (1986). The full curve is obtained from (3.1) using the solutions (2.12) and (2.13) of the advection–diffusion equation (1.1). The broken curve is obtained from (3.1) using the solutions (2.14) and (2.15) of the hyperbolic equation (1.2).

following values for $D(x)$ and $T(x)$:

$$D(x) = 0.027C(MU)\left(\frac{x}{M} + 19.3\right)^{-0.25} \tag{3.7}$$

$$T(x) = 0.182C\left(\frac{M}{U}\right)\left(\frac{x}{M} + 19.3\right)^{1.19}. \tag{3.8}$$

Now we can use (3.7) for $D(x)$, with $T = 0$, in (2.12) and (2.13) to compute $c_0(x)$ and $c_2(x)$ for the advection–diffusion equation. Then we use the results in (3.1) to obtain $\sigma^2(x)$. The integrals are evaluated numerically and the result is shown by the full curve in figure 2. Next we use both (3.7) and (3.8) in (2.14) and (2.15) to obtain c_0 and c_2 for the hyperbolic equation (1.2). Then (3.1) gives $\sigma^2(x)$, which is shown as the broken curve in figure 2. The value of $\alpha(x)/\beta(x)$ was checked and found to be positive.

It can be seen that the hyperbolic equation gives a much better fit to the data points in figure 2 than does the advection–diffusion equation, as was the case for the atmospheric diffusion experiments. The advection–diffusion equation predicts a plume width that is too wide close to the source. However, the agreement of the prediction of the hyperbolic equation with the experiment in the region near the source is not perfect. This might be due to the failure of the hyperbolic equation due to too rapid variation of $c(x, t)$ close to the source, in which case the full integrodifferential equation needs to be used. The discrepancy might also be due to experimental errors; the experimenters report that the region very close to the wire may be dominated by effects such as the wake of the heating wire itself or the increased viscosity of air due to the elevated temperature.

4. Derivation of an equation for the mean concentration.

Let $c\,(\boldsymbol{x}, t)$ be the concentration of a passive additive at the point $\boldsymbol{x}$ at time t in an incompressible fluid in turbulent motion with velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. We suppose that c satisfies the advection–diffusion equation

$$[\partial_t + u_i\partial_i - \partial_i K_{ij}\partial_j]c = g(\boldsymbol{x}, t). \tag{4.1}$$

Here K_{ij} is the molecular diffusion coefficient tensor, $\partial_i = \partial/\partial x_i$, g is the source strength distribution and repeated indices are summed. We assume that u and g are random functions, and then c will be random also. We denote the average of any function f by $\langle f\rangle$ and its fluctuating part by $f' = f - \langle f\rangle$, which we will also write as $f' = Pf$ in terms of a projection operator P. Our goal is to determine $\langle c\rangle$.

We begin by averaging (4.1) and writing $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \langle u_i\rangle\partial_i$, to obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j\right]\langle c\rangle + \langle u_i'\partial_i c'\rangle = \langle g\rangle. \tag{4.2}$$

Then we subtract (4.2) from (4.1) to obtain the following equation for c':

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j + Pu_i'\partial_i\right]c' = g' - u_i'\partial_i\langle c\rangle. \tag{4.3}$$

We write the solution of (4.3) as

$$c' = \left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j + Pu_i'\partial_i\right]^{-1}(g' - u_j'\partial_j\langle c\rangle) + c'(\boldsymbol{x}, 0). \tag{4.4}$$

Next we substitute (4.4) into (4.2) to obtain an equation for $\langle c\rangle$:

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j - \langle u_i'\partial_i\left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_k K_{km}\partial_m + Pu_k'\partial_k\right]^{-1}u_j'\rangle\partial_j\right\}\langle c\rangle$$
$$= \langle g\rangle - \left\langle u_k'\partial_k\left[\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j + Pu_i'\partial_i\right]^{-1}g'\right\rangle - \langle u_i'(\boldsymbol{x}, t)\partial_i\, c'(\boldsymbol{x}, 0)\rangle. \tag{4.5}$$

To estimate the terms in this operator we write $|\mathrm{d}\langle c\rangle/\mathrm{d}t| \sim \max(T^{-1}, |\langle\boldsymbol{u}\rangle|\,L^{-1})\langle c\rangle$ and $\left|u_k'\partial_k\langle c\rangle\right| \sim \langle\boldsymbol{u}'^2\rangle^{1/2}L^{-1}\langle c\rangle$. Here T and L are the time and length scales on which $\langle c\rangle$ varies, and we assume that the time and space derivatives do not cancel. The ratio of the second of these two terms to the first one is $\left|u_k'\partial_k\langle c\rangle\right| / |\mathrm{d}\langle c\rangle/\mathrm{d}t| \sim \langle\boldsymbol{u}'^2\rangle^{1/2}\min\left(\frac{T}{L}, \frac{1}{|\langle u\rangle|}\right)$. When $\langle c\rangle$ is independent of time then $T = \infty$, and this ratio becomes $\langle\boldsymbol{u}'^2\rangle^{1/2}/\,|\langle u\rangle|$. In the atmospheric measurements described in section 3.1, $|\langle u\rangle| = U$ and this ratio is 0.12, as is stated there at the end of the second paragraph. In the wind tunnel experiments described in section 3.2, $|\langle u\rangle| = U$ and (3.4) shows that this ratio satisfies $\langle\boldsymbol{u}'^2\rangle^{1/2}/U \leqslant \left[0.15(x_0/M)^{-1.43}\right]^{1/2} = \left[0.15(19.3)^{-1.43}\right]^{1/2} \approx 0.02$. Thus in both cases the term $u_k'\partial_k$ is small compared with $\mathrm{d}/\mathrm{d}t$.

When this is the case we shall neglect the u_k' term. The resulting operator $\mathrm{d}/\mathrm{d}t - \partial_k K_{km}\partial_m$ is then the diffusion operator, which can be inverted by using its Green function. For simplicity, we shall assume that the molecular diffusive part $\partial_k K_{km}\partial_m$ is small compared with the advective part $\mathrm{d}/\mathrm{d}t$, so we shall also neglect the molecular diffusive part.

With these assumptions, we just have to invert the first-order operator $\mathrm{d}/\mathrm{d}t$, i.e. to solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) \qquad \varphi(\boldsymbol{x}, 0) = 0. \tag{4.6}$$

To do so, we introduce the characteristic curve or particle path $\boldsymbol{y}(s, t, \boldsymbol{x})$ which passes through $\boldsymbol{x}$ at $s = t$. It is defined by

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}s} = \langle \boldsymbol{u}(\boldsymbol{y}, s)\rangle \qquad \boldsymbol{y}(t, t, \boldsymbol{x}) = \boldsymbol{x}. \tag{4.7}$$

In terms of $\boldsymbol{y}$ the solution φ of (4.6) is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-1} f = \int_0^t f\,[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\,\mathrm{d}s. \tag{4.8}$$

By using (4.8) for the inverse operator in (4.5) we obtain

$$(\mathrm{d}/\mathrm{d}t - \partial_i K_{ij}\partial_j)\langle c\rangle - \langle u_i'(\boldsymbol{x}, t)\partial_{x_i}\int_0^t u_j'[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\rangle \partial_{y_j}\langle c[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\rangle \mathrm{d}s$$
$$= \langle g\rangle - \langle u_i'(\boldsymbol{x}, t)\partial_{x_i}\int_0^t g'[\boldsymbol{y}(s, t, x), s]\,\mathrm{d}s\rangle - \langle u_i'(\boldsymbol{x}, t)\partial_i c'(\boldsymbol{x}, 0)\rangle. \tag{4.9}$$

We recall that $\partial_i u_i' = 0$. Therefore, we can rewrite the term in (4.9) involving u_i' and u_j' by introducing the correlation function

$$\mathcal{C}_{ij}(\boldsymbol{x}, t, \boldsymbol{y}, s) = \langle u_i'(\boldsymbol{x}, t)u_j'(\boldsymbol{y}, s)\rangle. \tag{4.10}$$

When u_i' is independent of g' and of $c'(\boldsymbol{x}, 0)$, for example, when g' and $c'(\boldsymbol{x}, 0)$ vanish, the last two terms in (4.9) vanish. Then we can write (4.9) in the form

$$(\mathrm{d}/\mathrm{d}t - \partial_i K_{ij}\partial_j)\langle c\rangle - \partial_{x_i}\int_0^t \mathcal{C}_{ij}[\boldsymbol{x}, t, \boldsymbol{y}(s, t, \boldsymbol{x}), s]\partial_{y_j}\langle c[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\rangle\,\mathrm{d}s = \langle g\rangle. \tag{4.11}$$

This is our main equation for $\langle c\rangle$. It involves the velocity correlation at points along the path of a particle moving with the mean velocity. The initial value of $\langle c\rangle$ is determined by the prescribed initial value of c. We note that $c'(\boldsymbol{x}, t)$ is given in terms of $\langle c\rangle$ by (4.4), in which the inverse operator can be approximated by that in (4.8).

As a first application of (4.11) we treat the case in which the correlation time and correlation length of the velocity fluctuations are small compared with the time scale and length scale on which $\langle c\rangle$ varies. Then we can replace $\partial_{y_j}\langle c\,[\boldsymbol{y}(s, t, \boldsymbol{x})s]\rangle$ by $\partial_{x_j}\langle c(\boldsymbol{x}, t)\rangle$ in the integrand in (4.11) and write the resulting equation in the form

$$\left[\partial_t + \langle u_i\rangle\partial_i - \partial_i\left[K_{ij} + D_{ij}(\boldsymbol{x}, t)\right]\partial_j\right]\langle c(\boldsymbol{x}, t)\rangle = \langle g\rangle. \tag{4.12}$$

Here $D_{ij}(x, t)$ is defined by the integral

$$D_{ij}(\boldsymbol{x}, t) = \int_0^t \mathcal{C}_{ij}\,[\boldsymbol{x}, t, \boldsymbol{y}(s, t, \boldsymbol{x}), s]\;\mathrm{d}s. \tag{4.13}$$

Thus in this case $\langle c\rangle$ satisfies an advection–diffusion equation, and the turbulent diffusion coefficient is given by (4.13). This expression for D_{ij} is similar to that derived by Taylor (1920), but here y is determined by the mean velocity rather than by the total velocity, and the upper limit is finite. When $K_{ij} = K\delta_{ij}$ and $D_{ij}(x, t) = D\delta_{ij}$, equation (4.12) becomes (1.1) in which the angular brackets are omitted.

Next we consider somewhat larger correlation times and lengths. We expand $\partial_{y_i}\langle c[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\rangle$ to first order in $t - s$, making use of (4.7), to obtain

$$\partial_{y_j}\langle c[\boldsymbol{y}(s, t, \boldsymbol{x}), s]\rangle = \partial_{x_j}\langle c(\boldsymbol{x}, t)\rangle - (t - s)[\langle u_k(\boldsymbol{x}, t)\rangle\partial_{x_j}\partial_{x_k}\langle c(\boldsymbol{x}, t) + \partial_{x_j}\partial_t\langle c(\boldsymbol{x}, t)\rangle] + \cdots. \tag{4.14}$$

We use (4.14) in the integrand in (4.11) and we get two integrals. The first integral is $D_{ij}(\boldsymbol{x}, t)$ and the second we call $T_{ij}(\boldsymbol{x}, t)D_{ij}(\boldsymbol{x}, t)$ with no summation:

$$T_{ij}(\boldsymbol{x}, t)D_{ij}(\boldsymbol{x}, t) = \int_0^t \mathcal{C}_{ij}[\boldsymbol{x}, t, \boldsymbol{y}(s, t, \boldsymbol{x}), s](t - s)\,\mathrm{d}s. \tag{4.15}$$

Thus T_{ij} is the correlation time corresponding to $\mathcal{C}_{ij}$, and (4.11) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \partial_i K_{ij}\partial_j\right)\langle c\rangle - \partial_i D_{ij}\partial_j\langle c\rangle + \partial_i T_{ij} D_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\partial_j\langle c\rangle = \langle g\rangle. \tag{4.16}$$

To eliminate the third derivatives of $\langle c\rangle$ from (4.16), we assume that $T_{ij} = T(\boldsymbol{x}, t)$ is independent of i and j. Then we use (4.12), neglecting K_{ij}, to obtain $\partial_i D_{ij}\partial_j\langle c\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle - \langle g\rangle$. We use this in the third derivative terms in (4.16) to replace them by second derivatives. After a bit of calculation we obtain the second-order equation

$$\left[T\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t} - \partial_i(K_{ij} + D_{ij})\partial_j\right]\langle c\rangle + \left[(\partial_i T)D_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\partial_j - T\partial_i\left(\frac{\mathrm{d}D_{ij}}{\mathrm{d}t}\right)\partial_j + T\partial_k D_{ij}(\partial_i u_k)\partial_j\right]\langle c\rangle = \langle g\rangle + T\frac{\mathrm{d}}{\mathrm{d}t}\langle g\rangle. \tag{4.17}$$

Here derivatives in parentheses apply only to the quantities within the parentheses. When $K_{ij} = K\delta_{ij}$ and $D_{ij} = D\delta_{ij}$, equation (4.17) reduces to (1.2) in which the angular brackets have been omitted.

When the flow is not incompressible $u_i\partial_i c$ in (4.1) must be replaced by $\partial_i(u_i c)$, and the analysis yields slightly different results. When molecular diffusion is not negligible, a more complicated inverse operator must be used in (4.5). This inverse appears in the resulting form of (4.11).

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